Journal of Mathematical Analysis and Applications **251,** 217–229 (2000)
doi:10.1006/jmaa.2000.7042, available online at http://www.idealibrary.com on IDEAL®

New Approximation Schemes for General Variational Inequalities

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Submitted by William F. Ames

Received April 5, 2000

In this paper, we suggest and consider a class of new three-step approximation schemes for general variational inequalities. Our results include Ishikawa and Mann iterations as special cases. We also study the convergence criteria of these schemes.

Key Words: variational inequalities; three-step schemes; convergence; fixed points.

1. INTRODUCTION

General variational inequalities, which were introduced and studied by Noor [10], are an important and useful generalization of variational inequalities. It has been shown that general variational inequalities provide us with a unified, simple, and natural framework to study a wide class of problems including unilateral, moving, obstacle, free, equilibrium, and economics arising in various areas of pure and applied sciences. Projection methods, Wiener–Hopf equations, and auxiliary principle techniques have been used to develop some efficient and powerful numerical methods for solving variational inequalities. In recent years, Noor [9, 12, 14] has suggested and analyzed some three-step forward-backward splitting algorithms for solving variational inequalities by using the updating techniques of the solution and auxiliary principle. These forward-backward splitting algorithms are similar to those of the θ-scheme of Glowinski and Le Tallec [7], which they suggested by using the Lagrangian technique. It is known that three-step schemes are versatile and efficient. These three-step schemes are a natural generalization of the splitting methods of Peaceman

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and Rachford [18] and Douglas and Rachford [4]. For applications of the splitting techniques to partial differential equations, see Ames [1] and the references therein. Inspired and motivated by the usefulness and applications of the splitting type methods, we suggest and analyze a new class of three-step approximation schemes for solving general variational inequalities and related problems. These new methods include the Mann and Ishikawa iterative schemes and modified forward-backward splitting methods of Tseng [19] and Noor [15] as special cases. Our results represent an improvement and refinement of the previously known results in these fields.

2. PRELIMINARIES

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot , \cdot \rangle$ and $\|\cdot\|$, respectively. Let K be a nonempty closed convex set in H.

For given nonlinear operators $T,\ g : H \to H$, consider the problem of finding $u \in H$, $g(u) \in K$ such that

$$\langle Tu, g(v) - g(u) \rangle \geq 0, \qquad \text{for all } g(v) \in K. \tag{2.1}$$

An inequality of type (2.1) is called a *general variational inequality*, which was introduced and studied by Noor [10] in 1988. It turned out that odd order and nonsymmetric obstacle, free, moving, unilateral, and equilibrium problems arising in various branches of pure and applied sciences can be studied via general variational inequalities.

We now list some examples.

1. For $g \equiv I$, the identity operator, the general variational inequality (2.1) collapses to finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0, \qquad \text{for all } v \in K, \tag{2.2}$$

which is called the standard variational inequality. For recent state-of-the-art applications, see [2, 3, 5–17, 19].

2. If $K^* = \{u \in H; \langle u, v \rangle \geq 0, \text{ for all } v \in K\}$ is a polar (dual) convex cone of a closed convex cone K in H, then problem (2.1) is equivalent to finding $u \in H$ such that

$$g(u) \in K, \qquad Tu \in K^*, \qquad \langle g(u), Tu \rangle = 0, \tag{2.3}$$

which is known as the general complementarity problem; see [3, 5, 16, 17]. If $g = I$, the identity operator, then problem (2.3) is called the generalized complementarity problem. For $g(u) = u - m(u)$, where m is a point-to-point mapping, then problem (2.3) is called the quasi(implicit) complementarity problem; see [16, 17] and the references therein.

3. If the convex set K depends upon the solution explicitly or implicitly, then the variational inequality (2.2) is called the quasi variational inequality. To be more precise, let $K(u)$ be a closed convex-valued set in H. Consider the problem of finding $u \in K(u)$ such that

$$\langle Tu, v - u \rangle \geq 0, \qquad \text{for all } v \in K(u). \tag{2.4}$$

Problem (2.4) is called the quasi variational inequality problem. For applications, numerical methods, and formulations, see [2, 5, 16]. It is known that in many important applications, the convex-valued $K(u)$ has the form

$$K(u) = m(u) + K, \tag{2.5}$$

where m is a point-to-point mapping and K is a closed convex set in H.

We now show that the quasi variational inequality is a special case of the general variational inequality (2.1). For this purpose, we need the following result.

LEMMA 2.1 [2]. *For a given $z \in H$, $u \in K$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \qquad \text{for all } v \in K, \tag{2.6}$$

if and only if

$$u = P_K z, \tag{2.7}$$

where P_K is a projection from H onto K and $\rho > 0$ is a constant. This property of the projection operator P_K plays an important part in obtaining our results.

Using Lemma 2.1, one can easily show that the quasi variational inequality (2.4) is equivalent to finding $u \in K(u)$ such that

$$\begin{aligned} u &= P_K(u)[u - \rho Tu] = P_{m(u)+K}[u - \rho Tu] \\ &= m(u) + P_K[u - m(u) - \rho Tu], \qquad \text{using (2.5)} \end{aligned}$$

which can be written as

$$g(u) = P_K[g(u) - \rho Tu], \tag{2.8}$$

where

$$g(u) = u - m(u). \tag{2.9}$$

It is known that the problem of finding the fixed point of (2.8) is equivalent to finding $u \in H$, $g(u) \in K$ such that (2.1) holds; see Lemma 3.1. Consequently, we conclude that the quasi variational inequality (2.4) with $K(u)$ defined by (2.5) is a special case of the general variational inequality (2.1).

4. We now show that the minimum of a class of differentiable nonconvex functions on the g-convex set K in H can be characterized by the general variational inequality (2.1). For this purpose, we recall the following well known concepts, which are mainly due to Youness [20].

DEFINITION 2.1. Let K be any set in H. The set K is said to be g-convex, if there exists a function $g : H \to H$ such that

$$g(u) + t(g(v) - g(u)) \in K, \qquad \text{for all } u, v \in K, t \in [0,1].$$

Note that every convex set is g-convex, but the converse is not true; see [20]. In passing, we remark that the notion of the g-convex set was introduced by Noor [10] implicitly in 1988.

From now onward, we assume that K is a g-convex set, unless otherwise specified.

DEFINITION 2.2. The function $F : K \to H$ is said to be g-convex, if

$$F(g(u) + t(g(v) - g(u))) \leq (1 - t)F(g(u)) + tF(g(v)),$$
$$\text{for all } u, v \in K, t \in [0,1].$$

Clearly every convex function is g-convex, but the converse is not true; see [20].

We now show that the minimum of a differentiable g-convex function on K in H can be characterized by the general variational inequality (2.1) and this is the main motivation of our next result.

LEMMA 2.2. *Let $F : K \to H$ be a differentiable g-convex function. Then $u \in K$ is the minimum of a g-convex function F on K if and only if $u \in K$ satisfies the inequality*

$$\langle F'(g(u)), g(v) - g(u) \rangle \geq 0, \qquad \text{for all } g(v) \in K, \tag{2.10}$$

where F' is the differential of F at $g(u)$.

Proof. Let $u \in K$ be a minimum of the g-convex function F on K. Then

$$F(g(u)) \leq F(g(v)), \qquad \text{for all } g(v) \in K. \tag{2.11}$$

Since K is a g-convex set, then for all $u, v \in K$, $t \in [0,1]$, $g(v_t) = g(u) + t(g(v) - g(u)) \in K$. Setting $g(v) = g(v_t)$ in (2.11), we have

$$\begin{aligned} F(g(u)) &\leq F(g(u) + t(g(v) - g(u))) \\ &\leq F(g(u)) + t(F(g(v) - g(u))). \end{aligned}$$

Dividing the above inequality by t and taking $t \to 0$, we have

$$\langle F'(g(u)), g(v) - g(u) \rangle \geq 0,$$

which is the required result (2.10).

Conversely, let $u \in K$, $g(u) \in K$ satisfy the inequality (2.10). Since F is a g-convex function, for all $u, v \in K$, $t \in [0,1]$, $g(u) + t(g(v) - g(u)) \in K$, and

$$F(g(u) + t(g(v) - g(u))) \leq (1-t)F(g(u)) + tF(g(v)),$$

which implies that

$$F(g(v)) - F(g(u)) \geq \frac{F(g(u) + t(g(v) - g(u))) - F(g(u))}{t}.$$

Letting $t \to 0$, we have

$$F(g(v)) - F(g(u)) \geq \langle F'(g(u)), g(v) - g(u) \rangle \geq 0, \qquad \text{using (2.10)},$$

which implies that

$$F(g(u)) \leq F(g(v)), \qquad \text{for all } g(v) \in K,$$

showing that $u \in K$ is the minimum of F on K in H.

Lemma 2.2 implies that the g-convex programming problem can be studied via the general variational inequality (2.1) with $Tu = F'(g(u))$. In a similar way, one can show that the general variational inequality is the Fritz–John condition of the inequality constrained optimization problem.

We also need the following concepts.

DEFINITION 2.3. For all $u, v \in H$, an operator $T: H \to H$ is said to be:

(i) *strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v \rangle \geq \alpha \|u - v\|^2$$

(ii) *Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leq \beta \|u - v\|.$$

From (i) and (ii), it follows that $\alpha \leq \beta$.

3. MAIN RESULTS

In this section, we suggest and analyze some new approximation schemes for solving the general monotone variational inequality (2.1). One can prove that the general variational inequality (2.1) is equivalent to the fixed point problem by invoking Lemma 2.1.

LEMMA 3.1 [10]. *The function $u \in H$ is a solution of the variational inequality* (2.1) *if and only if $u \in H$ satisfies the relation*

$$g(u) = P_K[g(u) - \rho Tu], \tag{3.1}$$

where P_K is the projection operator and $\rho > 0$ is a constant.

Lemma 3.1 implies that the general variational inequality (2.1) is equivalent to the fixed point problem. This alternative equivalent formulation is very useful from the numerical and theoretical points of view. The relation (3.1) can be written as

$$F(u) = u - g(u) + P_K[g(u) - \rho Tu]. \tag{3.2}$$

We now study those conditions under which the general variational inequality (2.1) has a unique solution and this is the main motivation of our next result.

THEOREM 3.1. *Let the operators $T, g : H \to H$ be both strongly monotone with constants $\alpha > 0$, $\sigma > 0$ and Lipschitz continuous with constants with $\beta > 0$, $\delta > 0$, respectively. If*

$$\left| \rho - \frac{\alpha}{\beta^2} \right| < \frac{\sqrt{\alpha^2 - \beta^2 k(2-k)}}{\beta^2}, \qquad \alpha > \beta\sqrt{k(2-k)}, k < 1, \tag{3.3}$$

where

$$k = 2\left(\sqrt{1 - 2\sigma + \delta^2}\right), \tag{3.4}$$

then there exists a unique solution $u \in H$, $g(u) \in K$ of the general variational inequality (2.1).

Proof. From Lemma 3.1, it follows that problems (3.2) and (2.1) are equivalent. Thus it is enough to show that the map $F(u)$ has a fixed point. For all $u, v \in H$,

$$\begin{aligned}
&\|F(u) - F(v)\| \\
&\quad = \|u - v - (g(u) - g(v)) + P_K[g(u) - \rho Tu] - P_K[g(v) - \rho Tv]\| \\
&\quad \leq \|u - v - (g(u) - g(v))\| \\
&\qquad + \|P_K[g(u) - \rho Tu] - P_K[g(v) - \rho Tv]\| \\
&\quad \leq 2\|u - v - (g(u) - g(v))\| + \|u - v - \rho(Tu - Tv)\|, \qquad (3.5)
\end{aligned}$$

where we have used the fact that the operator P_K is nonexpansive.

Since the operator T is strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, it follows that

$$\begin{aligned}
&\|u - v - \rho(Tu - Tv)\|^2 \\
&\quad \leq \|u - v\|^2 - 2\rho\langle Tu - Tv, u - v\rangle + \rho^2\|Tu - Tv\|^2 \\
&\quad \leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u - v\|^2. \qquad (3.6)
\end{aligned}$$

In a similar way, we have

$$\|u - v - (g(u) - g(v))\|^2 \leq (1 - 2\sigma + \delta^2)\|u - v\|^2, \qquad (3.7)$$

where $\sigma > 0$ and $\delta > 0$ are the strong monotonicity and Lipschitz continuity constants of the operator g.

From (3.5), (3.6), and (3.7), we have

$$\begin{aligned}
\|F(u) - F(v)\| &\leq \left(2\sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\right)\|u - v\| \\
&= (k + t(\rho))\|u - v\|, \qquad \text{using } (3.4). \\
&= \theta\|u - v\|, \qquad (3.8)
\end{aligned}$$

where

$$t(\rho) = \sqrt{1 - 2\alpha\rho + \rho^2\beta^2}. \qquad (3.9)$$

and

$$\theta = k + t(\rho). \qquad (3.10)$$

From (3.3), it follows that $\theta < 1$, which implies that the map $F(u)$ defined by has a fixed point, which is the unique solution of (2.1).

Now using the auxiliary principle technique, see Noor [12, 16], we can suggest the following predictor-corrector type algorithm for solving the general variational inequality (2.1).

ALGORITHM 3.1. For a given $u_0 \in H$, compute the approximate solution $\{u_n\}$ by the iterative schemes

$$\begin{aligned} \langle \rho T u_n + g(y_n) - g(u_n), g(v) - g(y_n) \rangle &\geq 0, \quad \text{for all } g(v) \in K \\ \langle \rho T y_n + g(w_n) - g(y_n), g(v) - g(w_n) \rangle &\geq 0, \quad \text{for all } g(v) \in K \\ \langle \rho T w_n + g(u_{n+1}) - g(w_n), g(v) - g(u_{n+1}) \rangle &\geq 0, \quad \text{for all } g(v) \in K, \\ & \qquad n = 0, 1, 2, \dots, \end{aligned}$$

where $\rho > 0$ is a constant.

Using Lemma 3.1, Algorithm 3.1 can be written in the equivalent form as:

ALGORITHM 3.2. For a given $u_0 \in H$, compute the approximate solutions $\{u_n\}$, $\{w_n\}$, and $\{y_n\}$ by the iterative schemes

$$\begin{aligned} g(y_n) &= P_K[g(u_n) - \rho T u_n] \\ g(w_n) &= P_K[g(y_n) - \rho T y_n] \\ g(u_{n+1}) &= P_K[g(w_n) - \rho T w_n], \qquad n = 0, 1, 2, \dots . \end{aligned}$$

If g is invertible, then Algorithm 3.2 can be written in the following form

ALGORITHM 3.3. For a given $u_0 \in H$, compute $\{u_n\}$ by the iterative scheme

$$g(u_{n+1}) = P_K[I - \rho T g^{-1}] P_K[I - \rho T g^{-1}] P_K[I - \rho T g^{-1}] g(u_n), \qquad n = 0, 1, 2, \dots .$$

Algorithm 3.3 is a three-step forward-backward splitting algorithm for solving general variational inequalities (2.1), which was suggested and analyzed by Noor [9] using the updating technique of the solution. This method is very similar to that of Glowinski and Le Tallec [7], which they suggested by using the Lagrangian technique. For the convergence analysis of Algorithms 3.1–3.3, see Noor [9].

Invoking Algorithm 3.2, we now suggest another three-step scheme for solving the general variational inequality (2.1).

ALGORITHM 3.4. For a given $u_0 \in H$, compute the approximate solution $\{u_n\}$ by the iterative schemes

$$y_n = (1 - \gamma_n)u_n + \gamma_n\{u_n - g(u_n) + P_K[g(u_n) - \rho T u_n]\} \quad (3.11)$$

$$w_n = (1 - \beta_n)u_n + \beta_n\{y_n - g(y_n) + P_K[g(y_n) - \rho T y_n]\} \quad (3.12)$$

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n\{w_n - g(w_n) + P_K[g(w_n) - \rho T w_n]\}, \quad n = 0, 1, 2, \ldots, \quad (3.13)$$

where $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$; for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n$ diverges.

For $\gamma_n = 0$, Algorithm 3.4 reduces to:

ALGORITHM 3.5. For a given $u_0 \in H$, compute $\{u_n\}$ by the iterative schemes

$$w_n = (1 - \beta_n)u_n + \beta_n\{u_n - g(u_n) + P_K[g(u_n) - \rho T u_n]\}$$

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n\{w_n - g(w_n) + P_K[g(w_n) - \rho T w_n]\}, \quad n = 0, 1, 2, \ldots,$$

which is known as the Ishikawa iterative scheme for the general variational inequality (2.1). Note that for $\gamma_n = 0$ and $\beta_n = 0$, Algorithm 3.4 is called the Mann iterative method.

For $g = I$, the identity operator, Algorithm 3.4 collapses to the following algorithm for the variational inequality (2.2), which appears to be a new one.

ALGORITHM 3.6. For a given $u_0 \in K$, compute $\{u_n\}$ by the iterative schemes

$$y_n = (1 - \gamma_n)u_n + \gamma_n P_K[u_n - \rho T u_n]$$

$$w_n = (1 - \beta_n)u_n + \beta_n P_K[y_n - \rho T y_n]$$

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n P_K[w_n - \rho T w_n], \qquad n = 0, 1, 2, \ldots .$$

where $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$; for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n$ diverges.

Now we suggest a perturbed iterative scheme for solving the general variational inequality (2.1).

ALGORITHM 3.7. For a given $u_0 \in H$, compute the approximate solution $\{u_n\}$ by the iterative schemes

$$\begin{aligned} y_n &= (1-\gamma_n)u_n + \gamma_n\{u_n - g(u_n) + P_{K_n}[g(u_n) - \rho T u_n]\} + \gamma_n h_n \\ w_n &= (1-\beta_n)u_n + \beta_n\{y_n - g(y_n) + P_{K_n}[g(y_n) - \rho T y_n]\} + \beta_n f_n \\ u_{n+1} &= (1-\alpha_n)u_n + \alpha_n\{w_n - g(w_n) P_{K_n}[g(w_n) - \rho T w_n]\} + \alpha_n e_n, \\ & \qquad n = 0, 1, 2, \ldots, \end{aligned}$$

where $\{e_n\}$, $\{f_n\}$, and $\{h_n\}$ are the sequences of the elements of H introduced to take into account possible inexact computations and P_{K_n} is the corresponding perturbed projection operator; and the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ satisfy $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$, for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

For $\gamma_n = 0$, we obtain the perturbed Ishikawa iterative method and for $\gamma_n = 0$ and $\beta_n = 0$, we obtain the perturbed Mann iterative schemes for solving the general variational inequality (2.1). If $g = I$, the identity operator, we obtain the perturbed iterative method for solving variational inequalities of type (2.2).

If $g = I$ and $K = H$, then Algorithm 3.7 is equivalent to the following three-step scheme for the nonlinear equations $Tu = 0$, which appears to be a new method.

ALGORITHM 3.8. For a given $u_0 \in H$, find the approximate solution $\{u_n\}$ by the iterative schemes

$$\begin{aligned} y_n &= (1-\gamma_n)u_n + \gamma_n T u_n + \gamma_n h_n \\ w_n &= (1-\beta_n)u_n + \beta_n T y_n + \beta_n f_n \\ u_{n+1} &= (1-\alpha_n)u_n + \alpha_n T w_n + \alpha_n e_n, \qquad n = 0, 1, 2, \ldots, \end{aligned}$$

where $\{e_n\}$, $\{f_n\}$, and $\{h_n\}$ are sequences of the elements of H introduced to take into account possible inexact computations and the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ satisfy $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$, for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

In brief, for a suitable and appropriate choice of the operators T, g and the space H, one can obtain a number of new and previously known iterative schemes for solving variational inequalities and related problems. This clearly shows that Algorithm 3.4 and Algorithm 3.7 are quite general and are unifying.

We now study the convergence criteria of Algorithm 3.4. In a similar way, one can analyze the convergence criteria of other algorithms.

THEOREM 3.2. *Let the operators* T, g *satisfy all the assumptions of Theorem* 3.1. *If the condition* (3.3) *holds*, *then the approximate solution* $\{u_n\}$ *obtained from Algorithm* 3.4 *converges to the exact solution* u *of the general variational inequality* (2.1) *strongly in* H *for* $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$; *for all* $n \geq 0$ *and* $\sum_{n=0}^{\infty} \alpha_n$ *diverges*.

Proof. From Theorem 3.1, we see that there exists a unique solution $u \in H$ of the general variational inequality (2.1). Let $u \in H$ be the unique solution of (2.1). Then, using Lemma 3.1, we have

$$u = (1 - \alpha_n)u + \alpha_n\{u - g(u) + P_K[g(u) - \rho Tu]\} \tag{3.14}$$

$$= (1 - \beta_n)u + \beta_n\{u - g(u) + P_K[g(u) - \rho Tu]\} \tag{3.15}$$

$$= (1 - \gamma_n)u + \gamma_n\{u - g(u) + P_K[g(u) - \rho Tu]\}. \tag{3.16}$$

From (3.11) and (3.14), we have

$$\begin{aligned}\|u_{n+1} - u\| &= \big\|(1 - \alpha_n)(u_n - u) + \alpha_n\big(w_n - u - \big(g(w_n) - g(u)\big)\big) \\ &\qquad + \alpha_n\big\{P_K[g(w_n) - \rho Tw_n] - P_K[g(u) - \rho Tu]\big\}\big\| \\ &\leq (1 - \alpha_n)\|u_n - u\| + 2\alpha_n\big\|w_n - u - \big(g(w_n) - g(u)\big)\big\| \\ &\qquad + \alpha_n\big\|w_n - u - \rho(Tw_n - Tu)\big\| \\ &\leq (1 - \alpha_n)\|u_n - u\| + \alpha_n(k + t(\rho))\|w_n - u\|, \\ &\qquad\qquad \text{using (3.4) and (3.9)}, \\ &= (1 - \alpha_n)\|u_n - u\| + \alpha_n\theta\|w_n - u\|, \qquad \text{using (3.10)}.\end{aligned} \tag{3.17}$$

In a similar way, from (3.12) and (3.15), we have

$$\begin{aligned}\|w_n - u\| &\leq (1 - \beta_n)\|u_n - u\| + 2\beta_n\theta\big\|y_n - u - \big(g(y_n) - g(u)\big)\big\| \\ &\qquad + \beta_n\big\|y_n - u - \rho(Ty_n - Tu)\big\| \\ &\leq (1 - \beta_n)\|u_n - u\| + \beta_n(k + t(\rho))\|y_n - u\|, \\ &\qquad\qquad \text{using (3.4) and (3.9)} \\ &\leq (1 - \beta_n)\|u_n - u\| + \beta_n\theta\|y_n - u\|, \qquad \text{using (3.10)},\end{aligned} \tag{3.18}$$

and from (3.13) and (3.16), we obtain

$$\begin{aligned}\|y_n - u\| &\leq (1 - \gamma_n)\|u_n - u\| + \gamma_n\theta\|u_n - u\|, \qquad \text{using (3.10)}, \\ &\leq (1 - (1 - \theta)\gamma_n)\|u_n - u\| \\ &\leq \|u_n - u\|.\end{aligned} \tag{3.19}$$

From (3.18) and (3.19), we obtain

$$\begin{aligned} \|w_n - u\| &\leq (1 - \beta_n)\|u_n - u\| + \beta_n \theta \|u_n - u\| \\ &= (1 - (1 - \theta)\beta_n)\|u_n - u\| \\ &\leq \|u_n - u\|. \end{aligned} \tag{3.20}$$

Combining (3.17) and (3.20), we have

$$\begin{aligned} \|u_{n+1} - u\| &\leq (1 - \alpha_n)\|u_n - u\| + \alpha_n \theta \|u_n - u\| \\ &= [1 - (1 - \theta)\alpha_n]\|u_n - u\| \\ &\leq \prod_{i=0}^{n} [1 - (1 - \theta)\alpha_i]\|u_0 - u\|. \end{aligned}$$

Since $\sum_{n=0}^{\infty} \alpha_n$ diverges and $1 - \theta > 0$, we have $\prod_{i=0}^{n}[1 - (1 - \theta)\alpha_i] = 0$. Consequently the sequence $\{u_n\}$ converges strongly to u. From (3.18) and (3.19), it follows that the sequences $\{y_n\}$ and $\{w_n\}$ also converge to u strongly in H. This completes the proof.

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